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Fermion structure of non-Abelian vortices in high density QCD

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We study the internal structure of a non-Abelian vortex in color superconductivity with respect to quark degrees of freedom. Stable non-Abelian vortices appear in the Color-Flavor-Locked phase whose symmetry $SU(3)_{\mathrm{c+L+R}}$ is further broken to $SU(2)_{\mathrm{c+L+R}} \otimes U(1)_{\mathrm{c+L+R}}$ at the vortex cores. Microscopic structure of vortices at scales shorter than the coherence length can be analyzed by the Bogoliubov-de Gennes (B-dG) equation (rather than the Ginzburg-Landau equation). We obtain quark spectra from the B-dG equation by treating the diquark gap having the vortex configuration as a background field. We find that there are massless modes (zero modes) well-localized around a vortex, in the triplet and singlet states of the unbroken symmetry $SU(2)_{\mathrm{c+L+R}} \otimes U(1)_{\mathrm{c+L+R}}$. The velocities v_i of the massless modes ($i = t, s$ for triplet and singlet) change at finite chemical potential $\mu \neq 0$, and decrease as μ becomes large. Therefore, low energy excitations in the vicinity of the vortices are effectively described by 1+1 dimensional massless fermions whose velocities are reduced $v_i < 1$.

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I. INTRODUCTION

The structure of vortices is crucial in understanding the dynamics of the phases in a broad subject of physics from condensed-matter physics to high-energy physics and astrophysics. Macroscopic dynamics of vortices is described by a spatially dependent order parameter for a certain broken symmetry. It is well described by the Ginzburg-Landau theory, which has conducted great progress in studies of vortices in superconductors at the length scales longer than the coherence length and the penetration depth. On the other hand, in order to understand short range structure of vortices such as the vortex core, one needs a microscopic theory in which vortices are described by the fermionic degrees of freedom. Such a self-consistent description including both the order parameter and fermions is known as the Bogoliubov-de Gennes (B-dG) equation [1]. The first investigation of the vortex structure by solving the B-dG equation was done in Ref. [2]. Fermions are trapped inside the vortex core if their energies are less than the gap of the ground state while they are scattering states if their energies are larger than it. Since this first study treated the order parameter (the superconducting gap) as a background field (and thus it is not self-consistent), successive researches were devoted towards finding the self-consistent solutions to the B-dG equation [3, 4]. The complete self-consistent description was achieved by considering both quasi-bound and scattering fermions in the vortex [5, 6]. These studies predicted an enhanced local density of fermion states at the Fermi level around the vortex core, which is experimentally observed in various materials [7, 8]. Recently the analysis of vortices in the B-dG equation has been also applied to the BEC-BCS crossover in fermionic cold atom systems [9, 10].

Among all fermion modes associated with vortices, zero modes (gapless fermions) play an especially important role in various subjects of physics. In non-relativistic theories including all condensed matter systems, fermion "zero modes" trapped in a vortex core naturally appear in the semi-classical approximation, but they are actually not necessarily exactly zero-energy states because they in general acquire small gaps $\sim \Delta^2/E_{\mathrm{F}}$ (called "minigaps") that are small enough compared with the gap Δ [11]. For instance, a conventional s-wave superconductor is such a case [2]. In chiral p-wave superconductors Majorana fermions trapped inside a vortex core have exactly zero energy, which attain non-Abelian statistics [12, 13] and give a candidate of quantum computations. Recently it has been found that even a conventional s-wave superconductor allows fermion zero modes in a vortex core when it is surrounded by a topological insulator [14]. Unlike non-relativistic theories, fermion zero modes in relativistic theories [15] have exactly zero energy, and the number of zero modes is determined by the index theorem to be $2n$ for n winding vortices [16]. Fermion zero modes are also important in the context of cosmic strings [17].

The study of vortices has been extended to color superconductivity in dense quark matter, where the attractive force between quarks induces a rich structure of symmetry breaking patterns [18]. At extremely high densities, a novel phase called the color-flavor-locked (CFL) phase will be realized, where up, down and strange quarks participate in the Cooper pairing with symmetry breaking pattern $SU(3)_{\mathrm{c}} \otimes SU(3)_{\mathrm{L}} \otimes SU(3)_{\mathrm{R}} \to SU(3)_{\mathrm{c+L+R}}$ [18–20]. The color-flavor structure of the gap in the CFL phase is given by $\Delta^{\alpha i} \propto \epsilon^{\alpha\beta\gamma}\epsilon^{ijk}\langle\psi^{\beta j}\psi^{\gamma k}\rangle$, where i, j, k are flavor indices, and α, β, γ are color indices. It was argued that superfluid vortices would exist as a result of breaking of

the global $U(1)_{\rm B}$ symmetry of the baryon number [21–23]. Color flux tubes have been suggested [22, 23] but they are topologically and dynamically unstable. The true ground states of vortices in the CFL phase are non-Abelian vortices found by Balachandran, Digal and Matsuura [24] which are topologically stable superfluid vortices carrying color magnetic fluxes inside their core. It is a 1/3 quantized vortex, namely it has 1/3 $U(1)_{\rm B}$ winding (circulation) and 1/9 tension of those of a global $U(1)_{\rm B}$ vortex [21–23], and it also has 1/3 amount of the color flux of that of a topologically and dynamically unstable color flux tube [22, 23]. The detailed gap profile function and the size of color flux has been calculated in Ref. [25] in the Ginzburg-Landau approach. One characteristic property of non-Abelian vortices is that the color-flavor-locked symmetry $SU(3)_{\rm c+L+R}$ of the ground state CFL phase is spontaneously broken down to its subgroup $SU(2)_{\rm c+L+R} \otimes U(1)_{\rm c+L+R}$ in the presence of a non-Abelian vortex. According to this symmetry breaking, Nambu-Goldstone zero modes appear around the vortex, which parametrize a coset space $\mathbf{C}P^2 \simeq SU(3)_{\rm c+L+R}/[SU(2)_{\rm c+L+R} \otimes U(1)_{\rm c+L+R}]$ [26, 27]. Therefore, the vortex solution allows a continuous family of solutions with degenerate energy, corresponding to $\mathbf{C}P^2$. These modes are called the internal orientational zero modes and each point of $\mathbf{C}P^2$ corresponds to the color flux which the non-Abelian vortex carries. Since this breaking occurs in the vicinities of the vortex, these modes are localized around it, and one can construct a 1+1 dimensional effective theory on the vortex world-sheet by integrating the original action over the vortex codimensions [28]. The interaction between two non-Abelian vortices at large distance is mediated by fluctuations of $U(1)_{\rm B}$ Nambu-Goldstone modes, yielding the universal repulsion between them irrespective to their $\mathbf{C}P^2$ orientations in the internal space [27, 29]. This predicts that each global $U(1)_{\rm B}$ vortex is dynamically unstable to decay into three non-Abelian vortices which have different color fluxes with the total flux cancelled out. Furthermore, it also predicts a non-Abelian vortex lattice at least when the lattice spacing is much larger than the vortex core size [27]. Such a vortex lattice is expected to be realized in a CFL quark matter core of a rapidly rotating neutron (or compact) star [30, 31].

However, these studies are all based on a macroscopic theory, namely the Ginzburg-Landau model which is valid only at the scale larger than the penetration depth and coherence length. In order to understand the whole structure of the non-Abelian vortices including the region inside their cores, we have to consider fermion dynamics from the B-dG equation. The most important fermion modes are the zero modes as mentioned which exist exactly at the Fermi energy. The purpose of the present paper is to study the fermion zero modes in a non-Abelian vortex, based on the B-dG equation.

This paper is organized as follows. In Sec. II we construct the fermion zero modes around a vortex background in the cases of a single flavor and the CFL phase. We find four fermion zero modes for a non-Abelian vortex in the CFL phase, which belong to the singlet and triplet representations of the unbroken symmetry $SU(2)_{\rm c+L+R}$. We find an interesting layer structure of the zero modes in the core of the vortices; the core size of the singlet zero mode is the half of the one of the triplet zero modes. In Sec. III we construct the 1+1 dimensional effective field theory of the fermion zero modes. We find that the velocity of the zero modes propagating along the vortex line depend on the chemical potential, and that the velocity of singlet is about quarter of that of the triplet at high baryon density. The last section is devoted to summary of the results and discussion of the future problems.

II. BOGOLIUBOV-DE GENNES EQUATION IN COLOR SUPERCONDUCTIVITY

A. Single flavor

As the simplest example which shares the essential physics with the CFL case, we first discuss a relativistic fermion system with a single flavor that exhibits superconductivity and allows for Abelian vortices. We treat a single (or, an isolated) vortex whose core is located on an infinite straight line along the z axis, and assume that the winding number (vorticity) is one [35]. As long as we treat the gap profile function $\Delta(r)$ as a background field, the description in the present subsection is very similar to that of the vortex-fermion system discussed in Refs. [15–17]. However, we consider here a fermionic matter at *finite densities* (otherwise color superconductivity does not take place) while the previous studies [15–17] were formulated only in the vacuum. As we will see later, the effects of finite densities are quite important for low energy excitations of the fermion modes in a vortex.

The Bogoliubov-de Gennes (B-dG) equation is useful when one considers inhomogeneous superconductivity [1]: in the presence of normal-super boundaries and impurities. Among such examples is the vortex in which the gap is locally reduced compared to the value in the homogeneous ground state. While the Ginzburg-Landau equation is usually used to find a vortex profile, it is valid only for spatial variations at scales longer than the coherent length. On the other hand, since the B-dG equation is represented by microscopic (fermion) degrees of freedom, it can in principle describe the internal structure of a vortex at shorter length scales. For the present case, it is expressed in the Nambu-Gor'kov (particle-hole) representation as

$$\mathcal{H}\Psi = \mathcal{E}\Psi \,, \tag{1}$$

with the Hamiltonian density in the mean-field approximation

$$\mathcal{H} = \begin{pmatrix} -i\gamma_0\vec{\gamma}\cdot\vec{\nabla}-\mu & \Delta(x)\gamma_0\gamma_5 \\ -\Delta^*(x)\gamma_0\gamma_5 & -i\gamma_0\vec{\gamma}\cdot\vec{\nabla}+\mu \end{pmatrix}. \tag{2}$$

Here, we consider a relativistic massless fermion and the energy is evaluated with respect to the Fermi surface (μ is the chemical potential of a fermion). The gap is supposed to take a vortex profile : $\Delta(x) = |\Delta(r)|\,\mathrm{e}^{i\theta}$ with $|\Delta(r=0)| = 0$ and $|\Delta(r=\infty)| = |\Delta|$ (where $r = \sqrt{x^2+y^2}$ and θ is the angle around the vortex).

As the first attempt to this problem, we treat the gap function as a background field as was done in Ref. [2]. However, strictly speaking, the B-dG equation (1) should be solved self-consistently because the gap profile function is expressed in terms of fermion degrees of freedom. Such a self-consistent description has been discussed in a simpler configuration like a kink, and recently it was achieved even for vortices in a standard type-II superconductor [5]. We leave such a description in the color superconductivity for future problems.

With the cylindrical shape of the vortex profile, one can write the eigenstates of Eq. (1) as $\Psi^{k_z}_{\pm,m}(r,\theta,z)$ with

$$\Psi^{k_z}_{\pm,m}(r,\theta,z) = \Psi_{\pm,m}(r,\theta)\,\mathrm{e}^{ik_z z}, \tag{3}$$

which has the momentum k_z in the z direction, and chirality $\pm$ corresponding to right and left. The Nambu-Gor'kov structure of $\Psi_{\pm,m}(r,\theta)$ is expressed as

$$\Psi_{\pm,m}(r,\theta) = \begin{pmatrix} \varphi_{\pm,m}(r,\theta) \\ \eta_{\mp,m-1}(r,\theta) \end{pmatrix}, \tag{4}$$

where particle $\varphi_{\pm,m}$ and hole $\eta_{\mp,m-1}$ components are eigenstates of the z component of the total spin operator J_z with their eigenvalues $m+1/2$ and $(m-1)+1/2$, respectively [15]. We note that the chiralities of a particle and a hole (defined by the eigenvalues of γ_5) are $\pm$ and $\mp$ for right and left modes, respectively.

Once the concrete form of the vortex profile $|\Delta(r)|$ is given, one can explicitly solve the B-dG equation (1). However, we are rather interested in low energy excitations which may be independent of the precise form of the profile. In fact, even without knowing the precise form of the profile, one can address the questions whether there exist zero modes or not, and if they exist, what are the properties of the zero modes. Note that these are highly nontrivial questions: in a homogeneous superconductor, even though the original fermion is massless, there is no excitation around the Fermi surface below the gap energy. Of course, the B-dG equation reduces to the standard Bogoliubov equation for a spatially constant gap, and there is no massless excitation in that case. On the other hand, since it is known that the vortex-fermion system *in the vacuum* allows for zero modes which are trapped in a vortex [15, 17], and that its existence is guaranteed from the topological considerations [16], one naturally expects that there exist zero-mode solutions *even at finite densities*. This is indeed the case. The B-dG equation at finite densities (1) for a generic profile function $|\Delta(r)|$ allows for zero-mode solutions, namely, solutions with the $\mathcal{E}=0$ eigenvalue. The zero modes are realized for $m=0$ and $k_z=0$, and the explicit forms can be written as, for the right mode ($+$ for a particle, $-$ for a hole)

$$\varphi_{+,0}(r,\theta) = C\,\mathrm{e}^{-\int_0^r|\Delta(r')|\,\mathrm{d}r'}\begin{pmatrix} J_0(\mu r) \\ iJ_1(\mu r)\,\mathrm{e}^{i\theta}\end{pmatrix}, \tag{5}$$

$$\eta_{-,-1}(r,\theta) = C\,\mathrm{e}^{-\int_0^r|\Delta(r')|\,\mathrm{d}r'}\begin{pmatrix} -J_1(\mu r)\,\mathrm{e}^{-i\theta} \\ iJ_0(\mu r)\end{pmatrix}, \tag{6}$$

and for the left mode ($-$ for a particle, $+$ for a hole)

$$\varphi_{-,0}(r,\theta) = C'\,\mathrm{e}^{-\int_0^r|\Delta(r')|\,\mathrm{d}r'}\begin{pmatrix} J_0(\mu r) \\ -iJ_1(\mu r)\,\mathrm{e}^{i\theta}\end{pmatrix}, \tag{7}$$

$$\eta_{+,-1}(r,\theta) = C'\,\mathrm{e}^{-\int_0^r|\Delta(r')|\,\mathrm{d}r'}\begin{pmatrix} J_1(\mu r)\,\mathrm{e}^{-i\theta} \\ iJ_0(\mu r)\end{pmatrix}, \tag{8}$$

where C and C' are normalization constants, and $J_n(x)$ is the Bessel function. We have represented the solutions in the Weyl (2-component) spinors. In obtaining the solutions above, we have imposed finiteness at $r=0$ and in the limit $r\to\infty$. When $\mu=0$, these solutions recover the zero-mode solutions in the vacuum found in Ref. [15], as they should. Notice that the zero-mode solutions above satisfy the following "Majorana-like" condition [36] ($\kappa=\pm1$):

$$\Psi = \kappa\,\mathcal{U}\,\Psi^*, \quad \mathcal{U} = \begin{pmatrix} 0 & \gamma_2 \\ \gamma_2 & 0 \end{pmatrix}, \tag{9}$$

which leads to $\mathcal{U}^{-1}\mathcal{H}\mathcal{U} = -\mathcal{H}^*$ and thus physically implies the equivalence between a particle and a hole. One can explicitly check that if the solution satisfies this condition, it indeed leads to $m = 0$, $k_z = 0$ and $\mathcal{E} = 0$.

In the limit of small size of the vortex, in other words, at a distance much away from the vortex, $|\Delta(r)|$ can be regarded as a constant $|\Delta|$ which is equal to the gap in the bulk space. Then, for example, the right mode is given by a simpler form

$$\varphi_{+,0}(r,\theta) \simeq C\,\mathrm{e}^{-|\Delta|r}\begin{pmatrix} J_0(\mu r) \\ iJ_1(\mu r)\,\mathrm{e}^{i\theta} \end{pmatrix}, \tag{10}$$

$$\eta_{-,-1}(r,\theta) \simeq C\,\mathrm{e}^{-|\Delta|r}\begin{pmatrix} -J_1(\mu r)\,\mathrm{e}^{-i\theta} \\ iJ_0(\mu r) \end{pmatrix}. \tag{11}$$

This asymptotic behavior clearly shows that the zero-mode solutions are well-localized around the vortex line.

B. Color-Flavor-Locked phase

1. *B-dG equation with a non-Abelian vortex*

Now we discuss the fermion structure of the non-Abelian vortex which appears in the CFL phase. The gap configuration of the non-Abelian vortex in the color-flavor representation $\Delta^{\alpha i} \propto \epsilon^{\alpha\beta\gamma}\epsilon^{ijk}\langle\psi^{\beta j \mathrm{T}} C\gamma_5\psi^{\gamma k}\rangle$ ($C = i\gamma^2\gamma^0$ is the charge conjugation, α, β, γ and i, j, k are the color and flavor indices, respectively) is given by [24]

$$\Delta(r,\theta) = \begin{pmatrix} \Delta_1(r,\theta) & 0 & 0 \\ 0 & \Delta_0(r) & 0 \\ 0 & 0 & \Delta_0(r) \end{pmatrix}, \tag{12}$$

where $\Delta_1(r,\theta) = |\Delta_1(r)|\,\mathrm{e}^{i\theta}$ corresponds to the vortex configuration with winding number one, and $\Delta_0(r)$ does not have a winding number (though it is not spatially constant). If two gaps are the same and constant $\Delta_1 = \Delta_0 = \Delta_{\mathrm{CFL}}$, it is the CFL phase which is symmetric under the rotation in $SU(3)_{\mathrm{c+L+R}}$. On the other hand, the gap structure (12) is invariant only under the transformation in $SU(2)_{\mathrm{c+L+R}} \otimes U(1)_{\mathrm{c+L+R}}$ which is a subgroup of $SU(3)_{\mathrm{c+L+R}}$. Still, the symmetry of the bulk CFL phase is restored[37] in the limit $r \to \infty$ because the gap profile function $|\Delta_1(r)|$ approaches the value of the CFL phase $|\Delta_1(r)| \to |\Delta_{\mathrm{CFL}}|$.

For the problem of our interest, it is sufficient to consider the following mean-field Hamiltonian density which exhibits the CFL phase:

$$\mathcal{H} = \psi_i^{\alpha\dagger}(-i\gamma_0\vec{\gamma}\cdot\vec{\nabla} - \mu)\psi_i^\alpha + \left[\Delta_{ij}^{\alpha\beta}(\psi_i^{\alpha\mathrm{T}} C\gamma_5\psi_j^\beta)^\dagger + \mathrm{h.c.}\right], \tag{13}$$

where the gap is now expressed as $\Delta_{ij}^{\alpha\beta} \propto \langle\psi_i^{\alpha\mathrm{T}} C\gamma_5\psi_j^\beta\rangle$.

An explicit representation of the B-dG equation $\mathcal{H}\Psi = \mathcal{E}\Psi$ is given by (for a similar representation in the homogeneous CFL phase, see Refs. [20, 32])

$$\begin{pmatrix}
\hat{\mathcal{H}}_0 & \hat{\Delta}_1 & \hat{\Delta}_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hat{\Delta}_1 & \hat{\mathcal{H}}_0 & \hat{\Delta}_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hat{\Delta}_0 & \hat{\Delta}_0 & \hat{\mathcal{H}}_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \hat{\mathcal{H}}_0 & -\hat{\Delta}_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\hat{\Delta}_1 & \hat{\mathcal{H}}_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \hat{\mathcal{H}}_0 & -\hat{\Delta}_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\hat{\Delta}_0 & \hat{\mathcal{H}}_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{\mathcal{H}}_0 & -\hat{\Delta}_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\hat{\Delta}_0 & \hat{\mathcal{H}}_0
\end{pmatrix}
\begin{pmatrix} u_r \\ d_g \\ s_b \\ d_r \\ u_g \\ s_r \\ u_b \\ s_g \\ d_b \end{pmatrix}
= \mathcal{E}
\begin{pmatrix} u_r \\ d_g \\ s_b \\ d_r \\ u_g \\ s_r \\ u_b \\ s_g \\ d_b \end{pmatrix}, \tag{14}$$

where we have introduced the notation e.g., u_r for the quark with color "red" and flavor "up" in the Nambu-Gor'kov representation. The matrices $\hat{\mathcal{H}}_0$ and $\hat{\Delta}_i$ ($i = 0$ and 1) are given as follows:

$$\hat{\mathcal{H}}_0 = \begin{pmatrix} -i\gamma_0\vec{\gamma}\cdot\vec{\nabla} - \mu & 0 \\ 0 & -i\gamma_0\vec{\gamma}\cdot\vec{\nabla} + \mu \end{pmatrix}, \tag{15}$$

$$\hat{\Delta}_i = \begin{pmatrix} 0 & \Delta_i\gamma_0\gamma_5 \\ -\Delta_i^\dagger\gamma_0\gamma_5 & 0 \end{pmatrix}. \tag{16}$$

2. Multiplets in $SU(2)_{\rm c+L+R}$

Eigenstates of Eq. (14) are classified into multiplets of the unbroken $SU(2)_{\rm c+L+R}$ symmetry: triplet, doublet and singlet states. Note first that nine quark states (3 flavors $\times$ 3 colors) can be decomposed by the generators of the $SU(3)_{\rm c+L+R}$ symmetry of the CFL phase together with a unit matrix [32]:

$$\begin{pmatrix} u_r & u_g & u_b \\ d_r & d_g & d_b \\ s_r & s_g & s_b \end{pmatrix} = \sum_{A=1}^{9} \Psi^{(A)} \frac{\lambda_A}{\sqrt{2}}\,, \tag{17}$$

where $\lambda_A\,(A = 1, \cdots, 8)$ are the Gell-Mann matrices normalized as ${\rm tr}(\lambda_A \lambda_B) = 2\delta_{AB}$ and $\lambda_9 = \sqrt{2/3}\cdot\mathbf{1}$. Thus, $A = 1, \cdots, 8$ components are the $SU(3)$ octet, and $A = 9$ the singlet. However, in the presence of a non-Abelian vortex, it does not make sense to mention the multiplets under the $SU(3)_{\rm c+L+R}$ transformation. We rather treat multiplets under the unbroken $SU(2)_{\rm c+L+R}$ rotation. The triplet and the singlet are respectively given by

$$\Psi_{\rm t} \equiv \Psi^{(1)}\lambda_1 + \Psi^{(2)}\lambda_2 + \Psi^{(3)}\lambda_3\,, \tag{18}$$

$$\Psi_{\rm s} \equiv \Psi^{(8)}\lambda_8 + \Psi^{(9)}\lambda_9\,. \tag{19}$$

By using the color-flavor representation, each component that appears above is expressed as $\Psi^{(1)} = (d_r + u_g)/\sqrt{2}$, $\Psi^{(2)} = (d_r - u_g)/(\sqrt{2}i)$, $\Psi^{(3)} = (u_r - d_g)/\sqrt{2}$ for the triplet and $\Psi^{(8)} = (u_r + d_g - 2s_b)/\sqrt{6}$, $\Psi^{(9)} = (u_r + d_g + s_b)/\sqrt{3}$ for the singlet. In addition to the triplet and singlet states, there are two 'doublet' states. However, we do not discuss them here because they do not contain zero modes. On the other hand, as we will see later, the triplet and singlet states include zero modes and are thus important.

One can explicitly check that $\Psi_{\rm t}$ and $\Psi_{\rm s}$ defined above indeed transform as a triplet and a singlet, respectively. The $SU(2)_{\rm c+L+R}$ rotation acts on the quark field Ψ as

$$\Psi \to \Psi' = U_F\, \Psi\, U_c^{\rm T} \tag{20}$$

where $U_F = {\rm e}^{i\vec{\theta}\cdot\vec{\lambda}/2}$ and $U_c = {\rm e}^{i\vec{\phi}\cdot\vec{\lambda}/2}$ are the $SU(2)_{\rm L/R}$ and $SU(2)_{\rm c}$ rotations, respectively. Since we used the vector $\vec{\lambda} = (\lambda_1, \cdots, \lambda_8)$, the parameters $\vec{\theta}$ and $\vec{\phi}$ are defined only for the first three components $\vec{\theta} = (\theta_1, \theta_2, \theta_3, 0, 0, 0, 0, 0)$ and $\vec{\phi} = (\phi_1, \phi_2, \phi_3, 0, 0, 0, 0, 0)$. For the $SU(2)$ color-flavor locking, the simplest choice is given by $\phi_1 = -\theta_1$, $\phi_2 = \theta_2$, and $\phi_3 = -\theta_3$. Note also that θ_i $(i = 1, 2, 3)$ are matrices in the Nambu-Gor'kov representation

$$\theta_1 = \tilde{\theta}_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \theta_2 = \tilde{\theta}_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \theta_3 = \tilde{\theta}_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{21}$$

with $\tilde{\theta}_i$ $(i = 1, 2, 3)$ being real numbers. Then, one finds that the infinitesimal changes of the triplet Ψ_t components are closed within the three components $\Psi^{(i)}$ $(i = 1, 2, 3)$:

$$\begin{aligned} \delta\Psi^{(1)} &= \theta_3\Psi^{(2)} - \theta_2\Psi^{(3)}, \\ \delta\Psi^{(2)} &= \theta_1\Psi^{(3)} - \theta_3\Psi^{(1)}, \\ \delta\Psi^{(3)} &= \theta_2\Psi^{(1)} - \theta_1\Psi^{(2)}, \end{aligned} \tag{22}$$

and the singlet Ψ_s is invariant

$$\delta\Psi^{(8)} = \delta\Psi^{(9)} = 0\,. \tag{23}$$

3. Zero-mode solutions

Apart from the complication coming from the color and flavor degrees of freedom, the structure of the B-dG equation is essentially the same as in the case of the single flavor fermion. Namely, requiring finiteness of the solutions with the help of the "Majorana-like" condition, one obtains the zero energy solutions both in the triplet and singlet states. The B-dG equation naturally yields both the left and right modes and the structure of the solutions are common in both cases as in the case of the single flavor. However, we discuss below only the right-hand modes of the zero modes firstly because the chiral symmetry is *indirectly* broken in the CFL phase (even though it is the $SU(2)$ CFL), and secondly because if the system couples to some external fields (as the edge states in the quantum hall effect) only one mode (left or right, depending on the properties of the external field) will remain as a zero mode.

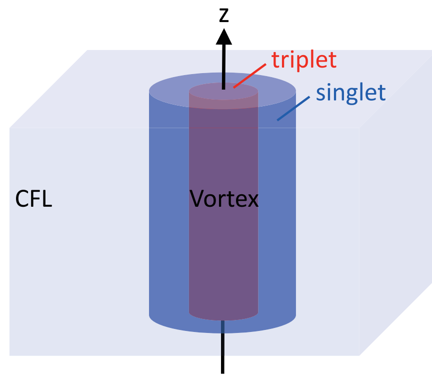

FIG. 1: Distributions of the triplet and singlet zero modes around a non-Abelian vortex.

The zero modes of the triplet right-handed quarks are analytically given by

$$\Psi^{(1)}(r,\theta) = C_1 \begin{pmatrix} \varphi(r,\theta) \\ \eta(r,\theta) \end{pmatrix}, \ \Psi^{(2)}(r,\theta) = C_2 \begin{pmatrix} \varphi(r,\theta) \\ -\eta(r,\theta) \end{pmatrix}, \ \Psi^{(3)}(r,\theta) = C_3 \begin{pmatrix} \varphi(r,\theta) \\ \eta(r,\theta) \end{pmatrix}, \tag{24}$$

where C_i are normalization constants and the particle (φ) and hole (η) components are

$$\varphi(r,\theta) = \mathrm{e}^{-\int_0^r |\Delta_1(r')|\mathrm{d}r'} \begin{pmatrix} J_0(\mu r) \\ iJ_1(\mu r)\,\mathrm{e}^{i\theta} \end{pmatrix}, \ \eta(r,\theta) = \mathrm{e}^{-\int_0^r |\Delta_1(r')|\mathrm{d}r'} \begin{pmatrix} -J_1(\mu r)\,\mathrm{e}^{-i\theta} \\ iJ_0(\mu r) \end{pmatrix}, \tag{25}$$

for generic shape of the vortex profile $|\Delta_1(r)|$. Notice that these solutions do not contain the unwinding gap $|\Delta_0(r)|$. This is easily understood from the structure of the B-dG equation (14). For example, if one looks at the d_r and u_g sector, the Hamiltonian contains only Δ_1. Another peculiarity about this solution is a minus sign of the hole component in $\Psi^{(2)}$. However, the minus sign must be there so that the zero-mode solutions in the triplet states transform as the adjoint representation under the $SU(2)_{\mathrm{c+L+R}}$ rotation as shown in Eq. (22).

On the other hand, the singlet zero modes depend on both $|\Delta_1(r)|$ and $|\Delta_0(r)|$, and due to this complication, we did not find explicit solutions for generic shape of the gaps. Still, approximate solutions are available where two gaps $|\Delta_1(r)|$ and $|\Delta_0(r)|$ are both replaced by the constant gap $|\Delta_{\mathrm{CFL}}|$ in the bulk CFL phase:

$$\Psi^{(8)}(r,\theta) = \frac{D}{\sqrt{6}} \begin{pmatrix} 2\varphi_1(r,\theta) + \varphi_2(r,\theta) \\ 2\eta_1(r,\theta) + \eta_2(r,\theta) \end{pmatrix}, \quad \Psi^{(9)}(r,\theta) = \frac{D}{\sqrt{3}} \begin{pmatrix} 2\varphi_1(r,\theta) - \frac{1}{2}\varphi_2(r,\theta) \\ 2\eta_1(r,\theta) - \frac{1}{2}\eta_2(r,\theta) \end{pmatrix}, \tag{26}$$

where D is a normalization constant and the particle and hole components are given by

$$\varphi_1(r,\theta) \simeq \mathrm{e}^{-\frac{|\Delta_{\mathrm{CFL}}|}{2}r} \begin{pmatrix} J_0(\mu r) \\ iJ_1(\mu r)\,\mathrm{e}^{i\theta} \end{pmatrix}, \ \eta_1(r,\theta) \simeq \mathrm{e}^{-\frac{|\Delta_{\mathrm{CFL}}|}{2}r} \begin{pmatrix} -J_1(\mu r)\,\mathrm{e}^{-i\theta} \\ iJ_0(\mu r) \end{pmatrix}, \tag{27}$$

$$\varphi_2(r,\theta) \simeq \mathrm{e}^{-\frac{|\Delta_{\mathrm{CFL}}|}{2}r} \begin{pmatrix} J_0(\mu r)\,\mathrm{e}^{-i\theta} \\ iJ_1(\mu r) \end{pmatrix}, \ \eta_2(r,\theta) \simeq \mathrm{e}^{-\frac{|\Delta_{\mathrm{CFL}}|}{2}r} \begin{pmatrix} -J_1(\mu r) \\ iJ_0(\mu r)\,e^{i\theta} \end{pmatrix}. \tag{28}$$

These approximate solutions are in fact asymptotic forms of the solutions valid at large distances much away from the vortex line.

In the triplet zero-mode solutions (25), if one approximates the gap profile $|\Delta_1|$ by the constant gap $|\Delta_{\mathrm{CFL}}|$ in the bulk CFL phase, one finds that the solutions decay as $\mathrm{e}^{-|\Delta_{\mathrm{CFL}}|r}$. On the other hand, the singlet zero modes decay as $\mathrm{e}^{-|\Delta_{\mathrm{CFL}}|r/2}$. Therefore, distribution of the singlet zero modes is wider than that of the triplet, as schematically shown in Fig. 1.

III. LOW ENERGY EFFECTIVE THEORY OF GAPLESS FERMIONS

In the previous section, we have discussed strictly zero-energy eigenstates of the B-dG equation, and found that such zero modes are well-localized on the plane perpendicular to the direction of the vortex line. While the transverse motion of such zero modes is frozen, they can move along the vortex line to give a linear dispersion with respect to the longitudinal momentum k_z. In this section, we develop an effective description of such low-energy excitations.

A. Single flavor

We again start with the single flavor case. We first notice that the Hamiltonian (2) can be decomposed into "transverse ($\perp$)" and "longitudinal (z)" parts:

$$\begin{aligned}\mathcal{H} &= \begin{pmatrix} -i\vec{\alpha}_\perp\cdot\vec{\nabla}_\perp - \mu & |\Delta|\mathrm{e}^{i\theta}\gamma_0\gamma_5 \\ -|\Delta|\mathrm{e}^{-i\theta}\gamma_0\gamma_5 & -i\vec{\alpha}_\perp\cdot\vec{\nabla}_\perp + \mu \end{pmatrix} + \begin{pmatrix} -i\alpha_z\frac{\partial}{\partial z} & 0 \\ 0 & -i\alpha_z\frac{\partial}{\partial z}\end{pmatrix} \\ &\equiv \mathcal{H}_\perp + \mathcal{H}_z. \end{aligned} \tag{29}$$

Since the longitudinal Hamiltonian $\mathcal{H}_z$ is linear with respect to the z derivative, it immediately implies that the low-energy excitations have linear dispersion relations with respect to k_z. Recalling that the zero-mode solutions $\Psi_0(r,\theta)$ (with $k_z = 0$) are in fact eigenstates of the transverse Hamiltonian $(\mathcal{H}_\perp + \mathcal{H}_z)\Psi_0(r,\theta) = \mathcal{H}_\perp\Psi_0(r,\theta) = 0$, one can see that the z dependence enters in a factorized way:

$$\Psi(t,z,r,\theta) = a(t,z)\Psi_0(r,\theta)\,, \tag{30}$$

where we have recovered time dependence and consider the Schrödinger equation $i\partial\Psi/\partial t = \mathcal{H}\Psi$. Note that the time dependence appears only in $a(t,z)$ because the zero-energy states are 'static'. Plugging Eq. (30) into the Schrödinger equation, and integrating it over transverse coordinates after multiplying it by the zero-mode solution $\Psi_0^\dagger$ from the left, we obtain the equation of motion for $a(t,z)$:

$$i\frac{\partial}{\partial t}a(t,z) = \int \Psi_0^\dagger(r,\theta)\mathcal{H}_z a(t,z)\Psi_0(r,\theta) r\mathrm{d}r\mathrm{d}\theta, \tag{31}$$

where we have used the normalization $\int \Psi_0^\dagger\Psi_0 r\mathrm{d}r\mathrm{d}\theta = 1$. We can rewrite this equation as

$$i\left\{\frac{\partial}{\partial t} + v(\mu,|\Delta|)\frac{\partial}{\partial z}\right\}a(t,z) = 0, \tag{32}$$

with the "velocity" $v(\mu,|\Delta|)$ defined by

$$v(\mu,|\Delta|) \equiv \int \Psi_0^\dagger(r,\theta)\begin{pmatrix}\alpha_z & 0 \\ 0 & \alpha_z\end{pmatrix}\Psi_0(r,\theta)\, r\mathrm{d}r\mathrm{d}\theta\,. \tag{33}$$

The solution to Eq. (32) is given by $a(t,z) \propto \mathrm{e}^{i\mathcal{E}t - ik_z z}$ with a linear (gapless) dispersion with respect to k_z:

$$\mathcal{E} = v(\mu,|\Delta|)\, k_z. \tag{34}$$

Therefore, low-energy excitations inside the vortex are gapless (massless) fermions described by Eqs. (32) and (33). One can indeed express these fermions in terms of spinors in 1+1 dimensions, and write down an equation similar to the Dirac equation.

One can compute the velocity by using the explicit form of the zero-mode solutions, i.e., the right mode (Eqs. (5), (6)) and the left mode (Eqs. (7), (8)). If we approximate the gap profile function $|\Delta(r)|$ by a constant $|\Delta|$, we obtain for the right mode

$$v_+(\mu,|\Delta|) = \frac{\mu^2}{|\Delta|^2+\mu^2}\frac{E(-\frac{\mu^2}{|\Delta|^2})}{E(-\frac{\mu^2}{|\Delta|^2}) - K(-\frac{\mu^2}{|\Delta|^2})} - 1\,, \tag{35}$$

and for the left mode

$$v_-(\mu,|\Delta|) = -v_+(\mu,|\Delta|)\,. \tag{36}$$

Here, $K(x)$ and $E(x)$ are the complete elliptic integrals of the first and second kinds. Therefore, the right (left) mode moves towards the plus (minus) direction of the z axis with the velocity less than the speed of light. In fact, as shown in Fig. 2, the velocity $v_+(\mu,|\Delta|)$ (35) is a decreasing function of $\mu/|\Delta|$, and $v_+ = 1$ (the speed of light) in vacuum ($\mu = 0$) [the result in vacuum agrees with that obtained by Witten [17] in the context of cosmic strings]. This is also seen from the asymptotic forms of the velocity: for small $\mu/|\Delta|$

$$v_+(\mu,|\Delta|) \simeq 1 - \frac{3}{4}\frac{\mu^2}{|\Delta|^2} + \mathcal{O}((\mu/|\Delta|)^4), \tag{37}$$

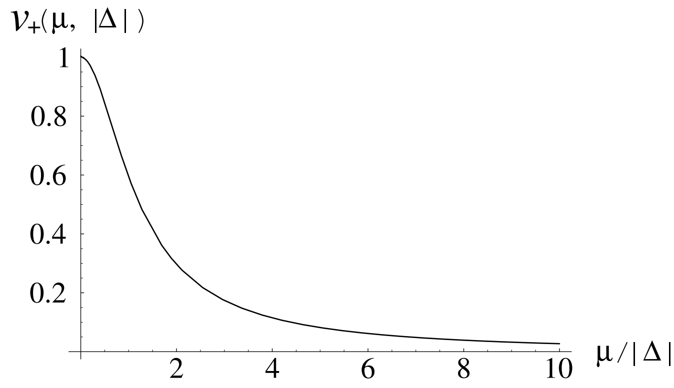

FIG. 2: Velocity of gapless fermions along the vortex axis.

while for large μ/Δ,

$$v_+(\mu, |\Delta|) \simeq \frac{|\Delta|^2}{\mu^2}\left(-1 + 2\ln 2 + \ln\frac{\mu}{|\Delta|}\right) + \mathcal{O}((|\Delta|/\mu)^4)\,. \tag{38}$$

Since the gap in the color superconductivity is parametrically given by $|\Delta(\mu)|/\mu \propto \mathrm{e}^{-b/g(\mu)}$ with b being a numerical constant and $g(\mu)$ the running coupling, $\mu/|\Delta|$ increases with increasing μ. Therefore, we conclude that the effective velocity $v_+(\mu)$ decreases with increasing μ. If we take typical values $\mu = 1000$ MeV and $|\Delta| = 100$ MeV, then the effective velocity is estimated as $v \simeq 0.027$ which is considerably smaller than the speed of light.

B. Color-Flavor-Locked phase

For the zero modes trapped in a non-Abelian vortex, one can perform the same procedure to obtain the effective theory. The low-energy effective theories of the left and right fermions in the triplet and singlet states are

$$i\left(\frac{\partial}{\partial t} + v_i^{\pm}\frac{\partial}{\partial z}\right) a_i(t,z) = 0\,, \tag{39}$$

with the dispersion relations

$$\mathcal{E} = v_i^{\pm}\, k_z, \tag{40}$$

where $i = t, s$ are for triplet and singlet, and $+, -$ are for the right and left modes. If we approximate the gap by a constant value in the bulk CFL phase, then the velocity of the triplet is exactly the same as the result of the single flavor (Eq. (35)) with the replacement of $|\Delta|$ by $|\Delta_{\mathrm{CFL}}|$, i.e.,

$$v_t^{\pm} = \pm v_+(\mu, |\Delta_{\mathrm{CFL}}|). \tag{41}$$

On the other hand, since the singlet zero modes have a wider transverse dependence, the velocity is evaluated by using the same function v_+ in Eq. (35), but with the replacement of $|\Delta|$ by $|\Delta_{\mathrm{CFL}}|/2$, i.e.,

$$v_s^{\pm} = \pm v_+(\mu, |\Delta_{\mathrm{CFL}}|/2). \tag{42}$$

In the vacuum $\mu = 0$, both the velocities of the triplet and the singlet are the same as the speed of light. However, with increasing densities, the two velocities start to deviate. For example, if one uses the asymptotic form of the velocity at large $\mu/|\Delta|$ (Eq. (38)), then one finds that the velocity of the singlet is about quarter of that of the triplet. Therefore, one can draw a schematic picture of the dispersion relations for all the $SU(2)$ multiplets as shown in Fig. 3.

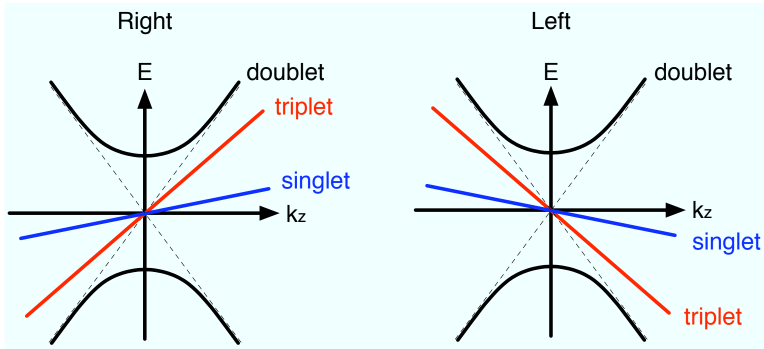

FIG. 3: Dispersion relations between energy E and momentum in z direction k_z for the ground states of triplet, singlet and doublet.

IV. SUMMARY AND DISCUSSION

We studied the fermion structure of a non-Abelian vortex in color superconductivity. By analyzing the Bogoliubov-de Gennes equation with the vortex gap profile, we found fermion zero modes at the Fermi level in the center of the vortex. We first demonstrated the single flavor case with an Abelian vortex as a warm-up before the more complicated case, and then turned to the CFL case with a non-Abelian vortex. We found that there are two gapless modes in the triplet and singlet states of $SU(2)_{\rm c+L+R} \otimes U(1)_{\rm c+L+R}$ which is the unbroken symmetry in the non-Abelian vortex. We also constructed the low-energy effective theory of the gapless fermions which describes the dynamics of fermions living on the vortex line (and thus in 1+1 dimensions). The gapless modes propagate along the vortex axis at the velocities smaller than the speed of light.

In this paper, we did not consider the presence of any gauge field in the vortex. However, the ordinary vortex of type-II superconductivity can trap a (quantized) magnetic flux in it. In the case of the non-Abelian vortex, there exist gauge fields which are associated with the unbroken $SU(2)_{\rm c+L+R} \otimes U(1)_{\rm c+L+R}$ symmetry. However, they are *massive* even in the vortex because only the winding component of gaps vanishes at the center of non-Abelian vortices, $\Delta(r=0) = {\rm diag}(0, *, *)$ with non-zero constants "$*$", where all generators of gauge transformations remain broken there. Thus they do not affect the low-energy excitations at the energy scale below the masses of gauge bosons. Therefore, we expect our effective theory of the gapless fermions is valid at least for very low-energy excitations.

In the discussion above, we assumed the limit of massless fermions for all flavors. In reality, the strange quark has a finite current mass, and cannot be regarded as a massless fermion. The effect of the strange quark mass on non-Abelian vortices has been studied recently in the Ginzburg-Landau model [33]. In the B-dG equation for vortices, we expect the energy-momentum dispersions of singlet and triplet zero modes are affected differently by this effect since the strange quark differently enters them (see Eqs. (18), (19)).

In the Ginzburg-Landau approach, a non-Abelian vortex has the $\mathbf{C}P^2$ *bosonic* zero modes depending on the flux as explained in Introduction [27]. A relation between those bosonic zero modes and fermionic zero modes in the B-dG equation studied in this paper is unclear at this stage. We expect that it will be clarified if we solve the B-dG equation self-consistently, which remains as an important future problem. It has also been studied [27] in the Ginzburg-Landau model that there exists a repulsive force between two parallel non-Abelian vortices, and that it does not depend on the $\mathbf{C}P^2$ zero modes if they are placed at a distance much larger than their core size. This situation is justified for instance in a vortex lattice with the lattice spacing larger than their core size. When a lattice spacing is comparable with the core size, we have to use the B-dG equation for multiple vortices. We expect that such an analysis gives a force depending on the $\mathbf{C}P^2$ zero modes, namely, on color fluxes which are carried by the vortices.

Note added: Just before we were about to finish the manuscript, we noticed through correspondences with Y. Nishida that he was doing very similar calculations as ours. It turned out from his recent work [34] that his main emphasis was on the topological aspects of the fermion zero modes, which we did not discuss in detail. Although his analysis is largely the same as ours in the single flavor case, he discussed only the Abelian vortices in the CFL phase, which is different from our analysis on the non-Abelian vortices.

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[35] Since the energy of a single vortex $\Delta = |\Delta|\,\mathrm{e}^{in\theta}$ with winding number $|n| \geq 2$ is proportional to n^2, it decays to n vortices each of which has a unit vorticity. Therefore, we consider only the case with $n = 1$. Note also that the anti-vortex with $n = -1$ gives qualitatively the same results.
[36] The Majorana condition with $\kappa = 1$ leads to the right mode, while $\kappa = -1$ the left mode.
[37] Precisely, the gap approaches $\Delta(r \to \infty) = \Delta_{\mathrm{CFL}}\mathrm{diag.}(\mathrm{e}^{i\theta}, 1, 1)$, and the extra phase in the first component can be factored out by a regular gauge transformation as $\Delta \to U\Delta = \Delta_{\mathrm{CFL}}\,\mathrm{e}^{i\theta/3}\mathrm{diag.}(1, 1, 1)$. Here $U(r, \theta)$ is a *regular* gauge transformation given by $U(r, \theta) = \exp[ig(r)(\theta/3)\mathrm{diag.}(-2, 1, 1)] \in SU(3)$ where $g(r)$ is an arbitrary function with the boundary conditions $g(r = 0) = 0$ and $g(r \to \infty) = 1$ [27].